On the decay of correlations in Sinai billiards with infinite horizon

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Abstract - We compute the decay of the autocorrelation function of the observable $|v_x|$ in the Sinai billiard and of the observable v_x in the associated Lorentz gas with an approximation due to Baladi, Eckmann and Ruelle. We consider the standard configuration where the disks is centered inside a unit square. The asymptotic decay is found to be $C(t) \sim c(R)/t$. An explicit expression is given for the prefactor $c(R)$ as a function of the radius of the scatterer. For the small scatterer case we also present expressions for the preasymptotic regime. Our findings are supported by numerical computations.

1 Introduction

In this paper we are going to present analytical and numerical evidence that typical correlation functions in a Sinai billiard with infinite horizon asymptotically decay as $C(t) \sim c(R)/t$. On the theoretical side there has been some heuristic argument [1, 2] and rigorous bounds [3] suggesting this type of behavior. We will strengthen the argument for the suggested type of decay and derive an expression for the constant $c(R)$ as a function of disk radius R. On the numerical side we investigate correlation functions via Monte Carlo simulations: a remarkably good agreement is found with theoretical expressions, which accurately predict the observed long-time behaviour, and also account for the main features of the preasymptotic region. The $1/t$ decay of the velocity autocorrelation function was, to our knowledge, first observed numerically in [1], and confirmed, in a somehow different approach, focusing on power spectrum behaviour, in [2]. More recently the asymptotic algebraic decay was detected in [4], while only the preasymptotic regime was investigated in the simulations described in [5].

We will focus on two particular correlation functions. First the autocorrelation function of the observable $|v_x|$ in the Sinai billiard. The modulus is taken to prevent the observable to

change sign by the frequent bounces on the square walls which would make our calculations more difficult. Secondly we study the autocorrelation of the observable v_x in the associated Lorentz gas, the difference from the former case is that the sign of the observable may now be changed by bounces on the disk(s).

The theoretical calculations will be carried out following an idea originally due to Baladi, Eckmann and Ruelle [6, 7, 8, 9, 10]. This approximation (called the BER approxmation) for intermittent systems divide the temporal evolution of the systems into *intervals*, each interval consist of one laminar and one chaotic interval. The approximation assumes the absence of correlations between disjoint intervals. In particular the lengths Δ_i are assumed mutually independend and one can define a probability distribtuion $p(\Delta)$, a function that encodes a lot of information about the system. For the observables considered in this paper the application of this approximation is almost trivial.

2 Application to the Sinai billiard

We will consider the version of the Sinai billiard [11] consisting of a unit square with a scattering disk, having radius R; $0 < 2R \leq 1$, centered on its midpoint. This is a system with infinite horizon, for any finite radius R there is a finite number of corridors, though which a particle may go without ever bounce onto a disk (these never-touching trajectories constitute a set of measure zero in phase space).

We let the point particle have unit velocity. The trajectory of a particle in the Sinai billiard consists of laminar intervals, (bouncing between the straight sections) interrupted by scatterings off the central disk. The variable Δ introduced above is simply the length of the trajectory *segment* between two disk bounces.

2.1 The distribution of reccurence times

We introduce the disk as surface of section with phase space coordinates x_s. We define $\Delta_s(x_s)$ as the traveling distance to the next disk bounce. The distribution $p(\Delta)$ is the distribution of recurrence times to the surface of section

$$p(\Delta) = \frac{1}{V_s} \int dx_s \; \delta(\Delta - \Delta_s(x_s)) \quad , \tag{1}$$

where $V_s = \int dx_s = 4\pi R$ using Birkhoff coordinates.

The relevant time scale is set by the average $< \Delta > \equiv \int \Delta p(\Delta) d\Delta$ which can be computed exactly as follows. When we integrate over phase space (or rather the energy surface) it is convenient to split up the phase space element according to $dx = dx_s d\tau$ where τ is is the coordinate along the the trajectory. The total phase space volume may now be expressed as

$$V = \int dx = \int dx_s \int_0^{\Delta_s(x_s)} d\tau = \int \Delta_s(x_s) dx_s = V_s < \Delta > = 4\pi R < \Delta > \quad . \tag{2}$$

On the other hand we have

$$V = \int dx \; dy \; dv_x dv_y \delta(1 - \sqrt{v_x^2 + v_y^2}) = (1 - \pi R^2) 2\pi \quad , \tag{3}$$

leading to the following expression for the expectation value of Δ

$$< \Delta > = \frac{1}{2R} - \frac{\pi R}{2} \quad . \tag{4}$$

In ref. [12] we derived the following expression for $p(\Delta)$ in the the small R limit

$$p_{R \to 0}(\Delta) = \begin{cases} \frac{12R}{\pi^2} & \xi < 1 \\ \frac{6R}{\pi^2 \xi^2} (2\xi + \xi(4 - 3\xi) \log(\xi) + 4(\xi - 1)^2 \log(\xi - 1) - (2 - \xi)^2 \log|2 - \xi|) & \xi > 1 \end{cases} \quad , \tag{5}$$

where $\xi = \Delta / < \Delta >$. Some smearing in ξ is required to make the limit well defined [12].

For finite R there is of course no such simple formula, but the tail ($\Delta \gg < \Delta >$) exhibit the asymptotic law

$$p(\Delta) \sim \frac{4\sigma(R)}{\pi \Delta^3} \quad , \tag{6}$$

where $\sigma(R)$ is the sum

$$\sigma(R) = \sum_{\mathbf{q} \in S} (2qR + \frac{1}{2qR} - 2) \quad . \tag{7}$$

where

$$S = \{\mathbf{q} = (n_x, n_y) | n_y > 0;\ n_x \geq 0;\ (n_x, n_y) = 1;\ q \equiv \sqrt{n_x^2 + n_y^2} < 1/2R\} \tag{8}$$

is the set of corridors.

The leading asymptotic behaviour of the function $\sigma(R)$ is in ref. [13] found to be $\sigma(R) \sim 1/(4\pi R^2)$ and in this limit we get

$$p(\Delta) \sim \frac{1}{\pi^2 R^2 \Delta^3} \quad R \to 0 \quad . \tag{9}$$

2.2 Correlation functions in the BER approixation

We will restrict ourselves to observables A that are changed only by bounces on the disk. The autocorrelation function is e.g. obtained as the time average

$$C_{AA}(t) = \langle A(t_0 + t) A(t_0) \rangle_{t_0} - \langle A \rangle^2 \quad . \tag{10}$$

The BER approximation assumes that there is no correlations if t and $t + t_0$ belongs to different intervals, that is $\langle A(t_0 + t) A(t_0) \rangle = \langle A^2 \rangle$ if there is no bounce between t and $t + t_0$ and $\langle A(t_0 + t) A(t_0) \rangle = \langle A \rangle^2$ otherwise.

Next let $P_0(t)$ denote the probability that the trajectory has *not* hit the disk between t and $t + t_0$. We can now write the correlation function in terms of conditional probabilities

$$C_{AA}(t) = P_0(t) \langle A^2 \rangle + (1 - P_0(t)) \langle A \rangle^2 - \langle A \rangle^2 = P_0(t) V(A) \quad , \tag{11}$$

where $V(A)$ is the variance of A.

The function $P_0(t)$ may be expressed in terms of $p(\Delta)$

$$P_0(t) = \frac{1}{\langle \Delta \rangle} \int_t^\infty \{ \int_u^\infty p(\Delta) d\Delta \} du \quad . \tag{12}$$

The $1/\Delta^3$ decay of $p(\Delta)$ thus implies a $1/t$ decay of the correlation function. From eq (6) we compute the tail of $P_0(t)$

$$P_0(t) \sim \frac{2\sigma(R)}{< \Delta > \pi} \frac{1}{t} \quad . \tag{13}$$

In the $R \to 0$ limit we get from eqs. (12), (9) and (4)

$$P_0(t) \sim \frac{1}{\pi^2 R} \frac{1}{t} \quad R \to 0 \quad . \tag{14}$$

For sufficiently small R one can use eq. (5) to obtain an expression for the entire $P_0(t)$ (the resulting expression is not particularly nice so we do not display it). For other radii R one can use a numerically obtained $p(\Delta)$ to compute $P_0(t)$.

First we will study the observable $A = |v_x|$. Since $\langle |v_x|^2 \rangle = 1/2$ and $\langle |v_x| \rangle = 2/\pi$ we get

$$C_{AA}(t) = P_0(t) (\frac{1}{2} - \frac{4}{\pi^2}) \quad , \tag{15}$$

2.3 Correlations in the Lorentz gas

A closely related problem is the decay of correlations in the associated Lorentz gas obtained by unfolding the bounded billiard into an infinite lattice of disks, cf. fig 1. If we kept considering the observable $A = |v_x|$ we would just recover the previous result. Let us instead consider the observable $B = v_x$. The difference is that the bounces on disks might now change the sign of B. The expression (11) may again be used, with the difference that $\langle B \rangle = 0$ so we get

$$C_{BB}(t) = P_0(t) \cdot \left(\langle B^2 \rangle - \langle B \rangle^2 \right) = P_0(t)/2 \quad . \tag{16}$$

It is well-known that this correlation function is related to the diffusion coefficient

$$D = \lim_{t \to \infty} \frac{1}{ft} < (\bar{x}(t) - \bar{x}(0))^2 > \quad , \tag{17}$$

via the Einstein-Green-Kubo formula [13]

$$D = \lim_{t \to \infty} \frac{2}{f} \int_0^t < \bar{v}(0) \cdot \bar{v}(s) > ds = \lim_{t \to \infty} 2 \int_0^t < v_x(0) \cdot v_x(s) > ds \tag{18}$$

where we have made use of the symmetry of our system $< v_x(0) \cdot v_x(s) >=< v_y(0) \cdot v_y(s) >$. f is the number of degrees of freedom.

Inserting eq. (16) into this formula we obtain the diverging diffusion constant

$$D(t) = \frac{2\sigma(R)}{< \Delta > \pi} \log t \quad , \tag{19}$$

in agreement with refs. [10, 13]

2.4 Comparison with numerical data

The numerical experiments are carried out through Monte Carlo phase integration (based on a subtractive random generator), using 10^7 initial conditions. Data become noisy when correlations decrease to the same order of magnitude as the integration error (which is inversely proportional to the square root of the number of initial conditions). It has been pointed out (see for example [14] and [5]) that in principle one might worry about errors due to exponential propagation of errors (due to positivity of the Lyapunov exponent and finite precision arithmetics): though our time sequences extend beyond the limit suggested by former arguments, the onset of power-law (asymptotic) decay is well within the allowed time scale (where the data are insensitive to statistical errors, which in turn dominate over the afore mentioned errors). We also remark that for hyperbolic billiards Lyapunov-induced errors did not seem to lead to detectable effects, see [15, 4].

In fig. 2 and 3 we show typical results of our numerical experiments: the numerical correlation function indeed exhibits a $1/t$ decay for long times. In the same figure we also plot theoretical curves obtained from eq. (15). The good agreement, even along the preasymptotic region, is suggestive of the applicability of BER approximation for such a system. From fig. 2 and 3 one can observe that BER expressions, like (15) are not able to reproduce preasymptotic oscillations in the correlation functions, while being quite close to their envelope, even for moderate time. Such oscillations are smaller and smaller in the $R \to 0$ limit. We comment on the deviation for large t in fig 3 in section 3.

In the small radius limit, eq. (5) suggests a scaling form of the correlation function $C_{AA,R}(t) = g(Rt)$ where g is independent of the value of R: in fig. 4 we show how the scaling is approximately correct for small times, and indeed is well confirmed in the asymptotic region.

Finally a comparison between correlation functions for A and B observables is presented in fig. 5.

3 Concluding remarks

There are two obvious ways of refining the present computations without abandoning the basic idea of the BER approximation.

There is a tacit assumption about isotropy behind the factorization $(1 - P_0(t))\langle A \rangle^2$ in eq. (11). It is true that the average of $|v_x|$ is exactly $= 2/\pi$ but only provided the average is taken over the entire phase space. But suppose we restrict the phase space integral to points lying on trajectories penetrating deep into the corridors then it is no longer exactly true that $\langle |v_x| \rangle = 2/\pi$ (except in the small R limit). This will lead to a small correction of the prefactor $c(R)$ of the asymptotic limit $C(t) \sim c(R)/t$ for observable A but not for observable B. Preliminary results indicate that this effect account for the deviation in the large t limit in fig. 3 and fig. 5.

Secondly, if we have a very long segment into one corridor, than there is an enhanced probability that the next segment goes into the same corridor [3]. Consecutives segments in the same corridor should be considered as belonging to the same interval as they are associated with nearly the same value of the observable. This effect will provide a small correction to $p(\Delta)$. We will return to these problems in future work.

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Figure captions

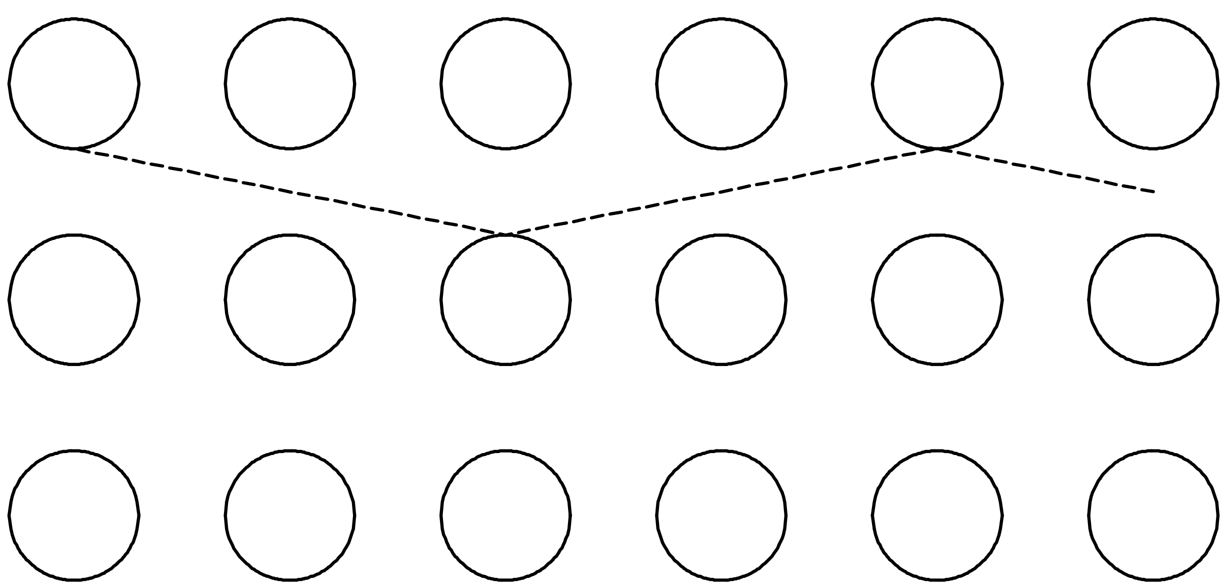
Figure 1: The Lorentz gas, obtained by unfolding the Sinai billiard over the plane.

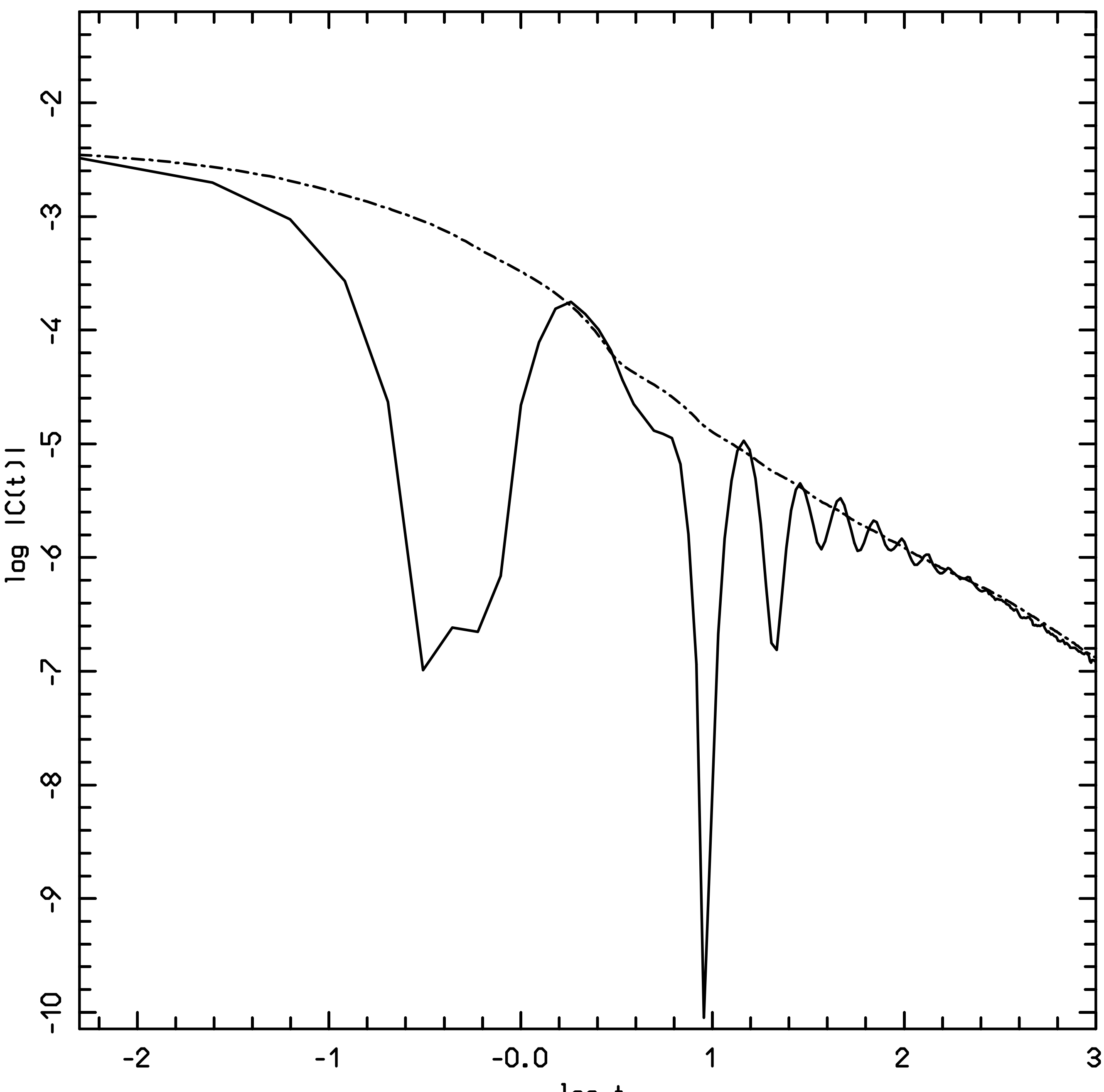
Figure 2: Experimental correlation function (full line), for $R = 0.318$. The dashed dotted line represents the BER approximation (15), using a numerical $p(\Delta)$.

Figure 3: The full line represents the numerical correlation function for $R = 0.106$, the dotted line is obtained from (15), using the theoretical small R limit of $p(\Delta)$ (5), while the dashed line has been obtained by using the asymptotic expression (13).

Figure 4: Correlation functions for different R values, plotted as functions of Rt. The dashed line corresponds to $R = 0.053$, the dashed dotted line to $R = 0.106$ and the full line to $R = 0.212$.

Figure 5: Numerical correlation function for observable B (upper full line) and A (lower full line) for the Lorentz gas (R=0.159). The dashed lines refer to the asymptotic expression (13) fed into equations (15) and (16).




log |C(t)|
-2
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-7
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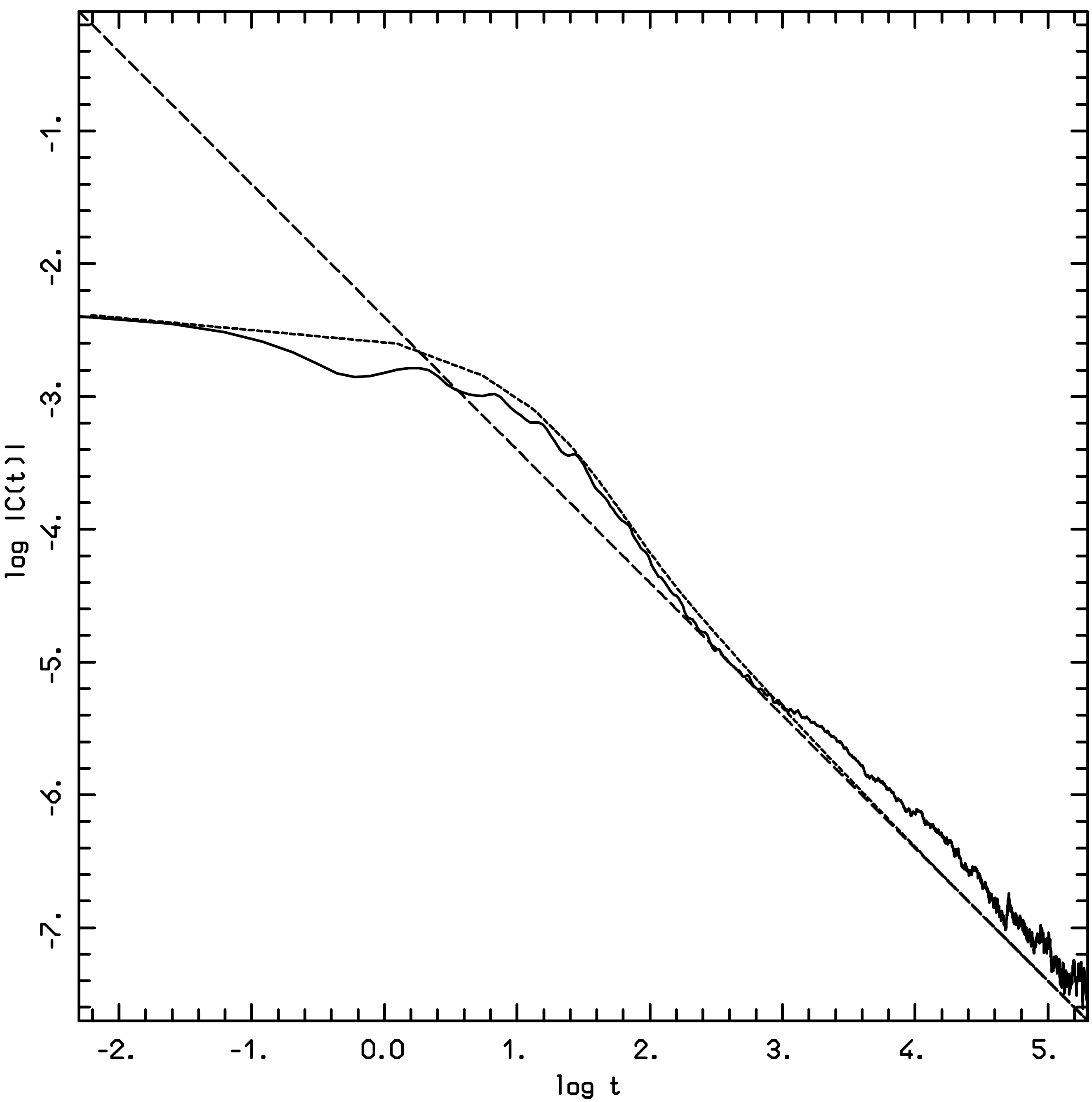
-9
-10
-2
-1
-0.0
1
2
3
log t



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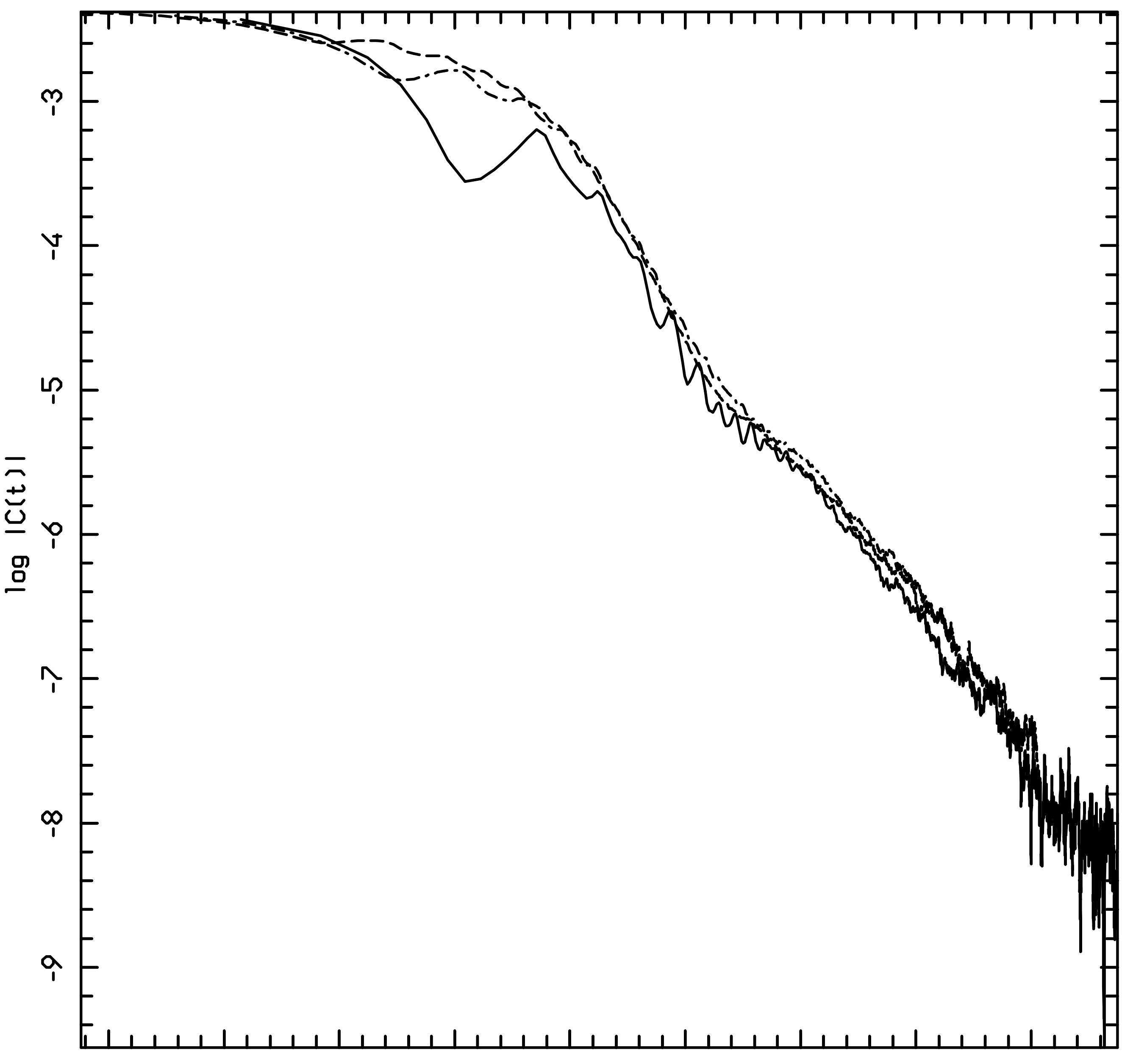

log |C(t)|
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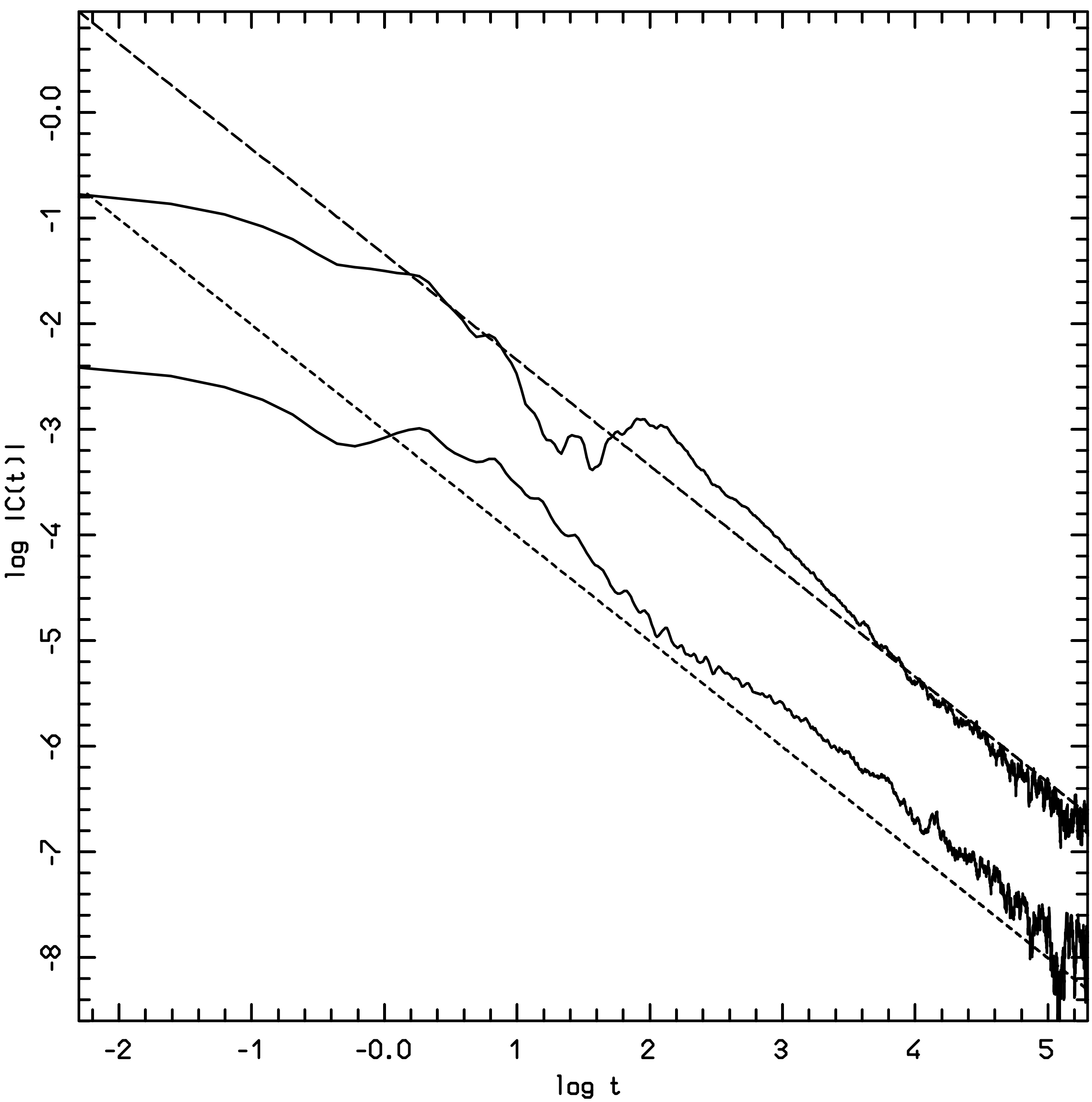
log(Rt)



-0.0
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log |C(t)|
-2
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log t